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Opinion attractiveness and its effect in opinion formation models

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ABSTRACT

In this work we introduce the idea of fitness in the context of opinion dynamics. We obtain the hydrodynamic equations for a kinetic model including the attractiveness or strength of each opinion, and we find a conserved quantity which enable us to characterize the consensus opinion. We compare this model with other rules of interaction, and we show interesting differences on the asymptotic behavior. Our results can help to explain how society could reach a consensus on opinions with lower fitness, despite of the fact that better options are available.

1. Introduction

Opinion dynamics is an interesting topic which has attracted a lot of attention in the last years. Depending on the context, the opinion space is discrete (cultural or religious norms, yes/no subjective questions, pro/contra tendencies on some topic, see [1–3]), or continuous (best location for a school or mall, degree of support to some measure, rating of a movie or a product, see [4–7]), and agent based models were used to study both cases. Let us remark that *opinion* is an abstraction representing different electoral parties or candidates, or the inclination to vote for some candidate [8,9], different products, locations in a city, and even the degree of a pixel in a gray scale, among many other examples.

According to sociologists, people interact and change opinions trying to fit in a group (social impact theory [10]), or modify their beliefs by interchanging of arguments (persuasive argument theory [11]). These processes can be thought as a sequence of microscopic interactions among agents, which generate observable macroscopic phenomena. Thus, ordinary and partial differential equations are useful tools in order to study these macroscopic properties, like magnetization [12,13], pattern formation [14,15], distribution of agents in the opinion space [16], mean opinions and its variance [17], among many others. These are ideas borrowed from statistical mechanics, and implemented by physicists and mathematicians, see [18–20].

However, the typical homogeneity assumptions on the molecules in a gas are not realistic, and heterogeneous agents were considered in several models. They have different characteristics: some of them are great speakers and persuade easily other agents while others are not able to persuade anyone; there are stubborns (or zealots) agents which never change opinions, and other agents have a low level of self-confidence or conviction and change opinion frequently, see for

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instance [21–23]. Biased, contrarians and hipster agents were considered too, see [1,24–26], which are resistant to follow the crowd and reject the opinion of the majority. In [27] we proved rigorously the long time behavior of a population in the space of opinions whenever each agent has its own degree of stubbornness and conviction. Recently in [28] the authors introduced the influence of the connectivity of different agents in a social media as a source of heterogeneity.

On the other hand, the heterogeneity of opinions or alternatives was less studied, despite its relevance in order to explain the outcome in the competition among several options. For instance, a classic example was the Betamax (Sony) and VHS (JVC) war, see [29,30]. It was won by the latter mainly by the different type of licensing, the consumer prices, and the recording times, despite the superior picture and sound quality of Betamax. Sony retained the rights to use Betamax technology, it was more expensive, and gives only one hour of recording; JVC offered a cheaper product which could record for two hours, the VHS technology was freely available for competitors, whom quickly transformed this format in the market standard annihilating their own projects like Philips VCR N1500, Avco Cartrivision, the Philips-Grundig Video 2000 or Sanyo V-Cord. This is an example of a competition between different products with different characteristics, and can be thought of as a competition between which product one thinks should buy.

So, in this work we introduce a continuous opinion model where the opinions are weighted by a coefficient representing its strength, attractiveness, or fitness. In order to fix ideas, we can think of the location of a new school in a city: each agent has some preferred location w (its opinion), and objectively there are places better than others, which defines a weight λ on the locations; the better is w, the greater is $\lambda(w)$. We assume that agents need to reach a consensus, or, in the language of sociologists, there exists a tendency to compromise.

In [5] the following interaction rule was introduced: whenever two agents with opinions w, w_* interact, the new opinions are given by

$$\begin{aligned} w' &= w + \gamma(w_* - w) \\ w'_* &= w_* + \gamma(w - w_*), \end{aligned}$$

where γ is a small parameter related to the strength of the interaction. This rule was widely used in opinion dynamics, since a Boltzmann type equation can be obtained as in classical kinetic theory.

Here, we modify this rule by including the attractiveness or fitness of w and w_* as follows

$$\begin{aligned} w' &= w + \gamma\lambda(w_*)(w_* - w) \\ w'_* &= w_* + \gamma\lambda(w)(w - w_*). \end{aligned} \tag{1}$$

Now, when two agents interact, the agent with an opinion of lower fitness will move more than the other agent. For simplicity, we do not consider here spontaneous changes of mind, which are usually modeled adding a random variable.

Let us mention the works [31–33], where two competing opinions, say ± 1, have an *attitude spectrum*, $A = \{\pm 1, \pm 2, \dots, \pm L\}$, which represent the strength of the opinion of an individual. Now, the behavior of agents depends on their attitudes according to some influence functions $\lambda(a)$ defined for $a \in A$. Following these works, we can interpret $\lambda(w_*)$ as the effort of an agent located at w_* to persuade the other agent, instead of an intrinsic characteristic of opinion w_*. So, we can consider this work as a generalization to a continuous attitude spectrum.

As we will show below, it is possible to derive the kinetic equations describing the temporal evolution of the distribution of agents in the space of opinions. Moreover, there exists a conserved quantity, preserved by the dynamics, and predetermines an opinion in terms of the initial datum which will be the consensus opinion reached by all the agents. However, an interesting symmetry breaking phenomena may appear if this opinion coincides with a zero of λ.

1.1. Organization of the paper

In Section 2 we define and analyze the microscopic rules of the model, and we derive the associated kinetic equations for the distribution of agents in the space of opinions. In Section 3 we describe the theoretical results and we find a conserved quantity which characterizes the consensus opinion. In Section 4 we present the agent based simulations. We conclude in Section 5.

2. Kinetic model

2.1. Microscopic rules

We will represent the space of alternatives or opinions by the interval $[-1, 1]$, and we assume that there exists a function $\lambda : [-1, 1] \to \mathbb{R}_{\geq 0}$ representing the fitness or attractiveness of each opinion.

It is reasonable to consider continuous functions with a finite number of jump discontinuities. The continuity implies that small changes of opinion do not represent an abrupt change of fitness, except for the case when the opinion crosses a jump discontinuity, which can be thought as crossing some threshold. For instance, one can support some political party or a football club, and at some point becomes an affiliated member of the party or the club.

Let us consider a society of $N \gg 1$ agents, and they will interact randomly in pairs. Given two agents with opinions w, w_*, the opinions update after an interaction are given by

$$\begin{aligned} w' &= w + \gamma\lambda(w_*)(w_* - w) \\ w'_* &= w_* + \gamma\lambda(w)(w - w_*). \end{aligned}$$

Here, $\gamma \leq 1/(2\max\lambda)$ is fixed and regulates the strength of the interaction independently of the fitness of the opinions involved. On the other hand, the higher is the fitness of opinion w_*, the larger is the change from w to w'. Observe that unattractive opinions with $\lambda \approx 0$, do not generate a great impact on the final outcome of the interaction.

The opinion update is well defined if $w', w'_* \in [-1, 1]$, thus we need to impose a bound on λ. From $\gamma < 1/2$, if we assume that $0 \leq \lambda \leq 1$, then the condition $|w'|, |w'_*| \leq 1$ is fulfilled,

$$\begin{aligned} |w'| &= |w + \gamma\lambda(w_*)(w_* - w)| \\ &\leq \gamma|1 - \lambda(w_*)||w| + \gamma\lambda(w_*)|w_*| \\ &\leq \frac{1}{2}(|1 - \lambda(w_*)| + \lambda(w_*)). \end{aligned}$$

The mean opinion after an interaction is not conserved unless $\lambda(w) = \lambda(w_*)$, since

$$w' + w'_* = w + w_* + \gamma[\lambda(w) - \lambda(w_*)](w - w_*).$$

2.2. Kinetic equations

In order to characterize the consensus opinion it is convenient to derive and analyze a kinetic equation describing the evolution of the density of agents in the space of opinions.

Let $w_i(t)$ be the opinion of agent i at time t, and let us define the empirical distribution of agents $f_N(w, t)$,

$$f_N(w, t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{w - w_i(t)},$$

where δ is a Dirac mass at 0.

We assume that the interactions occur at rate 1, so the expected number of interactions in a time interval $(t, t + dt)$ is dt, and hence the expected rate of change of w_i is

$$\begin{aligned} \frac{w_i(t + dt) - w_i(t)}{dt} &= \frac{1}{N}\sum_{j=1}^{N} \gamma\lambda(w_j)(w_j - w_i) \\ &= \int_{-1}^{1} \gamma\lambda(w)(w - w_i)df_N(w, t). \end{aligned}$$

By considering an auxiliary smooth test function $\varphi : [-1, 1] \to \mathbb{R}$, we define the observable $\int \varphi df_N$, the mean value of φ at time t. Its time evolution is determined by

$$\begin{aligned} \frac{d}{dt}\int_{-1}^{1} \varphi(w)df_N(w, t) &= \frac{1}{N}\frac{d}{dt}\sum_{i=1}^{N} \varphi(w_i) \\ &= \frac{1}{N}\sum_{i=1}^{N} \frac{d\varphi(w_i)}{dw_i}\frac{dw_i}{dt} \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{d\varphi(w_i)}{dw_i}\gamma\lambda(w_j)(w_j - w_i) \\ &= \gamma\int_{-1}^{1}\int_{-1}^{1} \frac{d\varphi(w)}{dw}\lambda(w_*)(w_* - w)df_N(w, t)df_N(w_*, t) \\ &= \gamma\int_{-1}^{1} \frac{d\varphi(w)}{dw}\Big[\int_{-1}^{1} \lambda(w_*)(w_* - w)df_N(w_*, t)\Big]df_N(w, t). \end{aligned}$$

Given any probability measure df on $[-1, 1]$, let us define the mean value of a function g as

$$\langle g \rangle = \int_{-1}^{1} g(w)df(w).$$

Hence, we get

$$\int_{-1}^{1} \lambda(w_*)(w_* - w)df_N(w_*, t) = \int_{-1}^{1} w_*\lambda(w_*)df_N(w_*, t) - \int_{-1}^{1} w\lambda(w_*)df_N(w_*, t)$$
$$= \langle w\lambda(w)\rangle - \langle\lambda\rangle w,$$

since f_N is a probability distribution and thus its total mass equals one.

By taking the limit $N \to \infty$, we denote by f the corresponding density for infinitely many agents, and the limit of the kinetic equation obtained before,

$$\frac{d}{dt}\int_{-1}^{1} \varphi(w)df = \gamma \int_{-1}^{1} \frac{d\varphi(w)}{dw}\Big[\langle w\lambda(w)\rangle - \langle\lambda\rangle w\Big]df \tag{2}$$

is the weak formulation of the transport equation

$$\frac{\partial}{\partial t}f(w,t) + \frac{\partial}{\partial w}\left(\gamma\Big[\langle w\lambda(w)\rangle - \langle\lambda\rangle w\Big]f(w,t)\right) = 0. \tag{3}$$

Let us stress that both $\langle w\lambda(w)\rangle$ and $\langle\lambda\rangle$ evolve with time, since they are computed with respect to $f(w,t)$.

2.3. A different interaction rule

In several works, see for instance [4,17], and [34] for a rigorous analysis, a related dynamic was considered,

$$\begin{aligned} w' &= w + \gamma P(w)(w_* - w) \\ w'_* &= w_* + \gamma P(w_*)(w - w_*). \end{aligned} \tag{4}$$

Here, each agent is influenced by its own opinion, and assuming that P is nonincreasing on $|w|$, we interpret it as a resistance to change for agents with more extreme opinions.

We can repeat the previous computations for the microscopic rule (4), and we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{-1}^{1} \varphi(w)df_N(w,t) \\ &= \gamma \int_{-1}^{1} \frac{d\varphi(w)}{dw}P(w)\Big[\int_{-1}^{1}(w_* - w)df_N(w_*, t)\Big]df_N(w,t) \\ &= \gamma \int_{-1}^{1} \frac{d\varphi(w)}{dw}P(w)(\langle w\rangle - w)df_N(w,t). \end{aligned}$$

The corresponding transport equation in the limit of infinitely many agents is

$$\frac{\partial}{\partial t}f(w,t) + \frac{\partial}{\partial w}\left(\gamma P(w)[\langle w\rangle - w]f(w,t)\right) = 0. \tag{5}$$

Despite the similarity of both Eqs. (3) and (5), there are important differences in the long time behavior of their solutions. We will perform some simulations illustrating the dissemblance between them in Section 4.

3. Theoretical results

3.1. Theoretical analysis

First, we can check that the dynamics (1) preserves the total mass of agents by replacing $\varphi \equiv 1$ in Eq. (2). We have $d\varphi/dw = 0$, and therefore we get

$$\frac{d}{dt}\int_{-1}^{1} df(w,t) = 0.$$

Now, let us call $m(t)$ the mean opinion,

$$m(t) := \langle w\rangle = \int_{-1}^{1} w\,df(w,t),$$

and let us analyze its behavior. By using $\varphi(w) = w$ as a test function in (2), we obtain that the evolution of the expected value of $m(t)$ is given by

$$\frac{d\langle m(t)\rangle}{dt} = \gamma[\langle w\lambda(w)\rangle - \langle\lambda\rangle m(t)]. \tag{6}$$

Notice that the mean opinion is not conserved, as suggested by the microscopic rule of interaction, and hence we cannot expect that $m(t)$ remains constant except for some particular cases, for instance, when λ is constant.

Denote as $f_\infty(w) = \lim_{t\to\infty} f(w,t)$ and $m_\infty = \lim_{t\to\infty} m(t)$, whenever these limits exist. In particular, if f_∞ exists, then $m_\infty = \int_{-1}^{1} w f_\infty(w)$.

However, observe that $f_\infty = \delta_a$ is an equilibrium for any value of a, being a the corresponding value for m_∞. Indeed, when taking mean value with respect to $f_\infty = \delta_a$ we have $\langle\lambda\rangle = \lambda(a)$ and $\langle w\lambda\rangle = a\lambda(a)$. Therefore, this is not enough to characterize the asymptotic behavior of the mean opinion.

3.2. Long time behavior

In order to study the long time behavior of $f(w,t)$ we need to find some conserved quantity. Let Λ be an antiderivative of λ, and let us show that $\int_{-1}^{1} \Lambda(w)df(w,t)$ is preserved by the evolution.

By taking $\varphi = \Lambda$ in Eq. (2), we get

$$\begin{aligned}
\frac{d}{dt}\int_{-1}^{1} \Lambda(w)df &= \gamma \int_{-1}^{1} \frac{d\Lambda(w)}{dw}\Big[\langle w\lambda(w)\rangle - \langle\lambda\rangle w\Big]df \\
&= \gamma \int_{-1}^{1} \lambda(w)\Big[\langle w\lambda(w)\rangle - \langle\lambda\rangle w\Big]df \\
&= \gamma[\langle\lambda\rangle\langle w\lambda(w)\rangle - \langle\lambda\rangle\langle w\lambda(w)\rangle] \\
&= 0,
\end{aligned}$$

and consequently

$$\int_{-1}^{1} \Lambda(w)df(w,t) = \int_{-1}^{1} \Lambda(w)df(w,0).$$

Accordingly, assuming that $f_t \to \delta_{m_\infty}$, we have $\Lambda(m_\infty) = \int_{-1}^{1} \Lambda(w)df(w,0)$, which implies

$$m_\infty = \Lambda^{-1}\Big(\int_{-1}^{1} \Lambda(w)df(w,0)\Big). \tag{7}$$

The convergence $f(w,t) \to \delta_{m_\infty}$ can be expected, since the dynamics is contractive. Intuitively, the rightist (respectively, leftist) agent can only move to the left (resp., right), and cannot go further away. However, a rigorous proof of this fact is out of the scope of this work.

Remark 3.1. Let us stress that random fluctuations in the simulations can change m_∞, and thus different limiting values can appear for each realization, even when we start with the same initial conditions. This value is expected to hold in mean.

However, when λ has a zero of high multiplicity at some point a, $\Lambda(w)$ increases slowly near a, and it is almost a constant function. This implies that, if in addition

$$a = \int_{-1}^{1} \Lambda(w)df(w,0),$$

random fluctuations are amplified when we take $\Lambda^{-1}(a)$.

4. Agent based simulations

We have performed several agent based simulations in Octave. We start with $N = 1000$ agents randomly distributed in $[-1, 1]$ with an uniform distribution, except for a few test cases with other distributions, not included here. Different values of N were considered, larger ones, up to $N = 10^5$, to confirm the long time behavior of agents distributions.

The parameter γ in the interaction was fixed as $\gamma = 0.005$. Let us note that in Eqs. (2) and (6) we can get rid of it by rescaling time. As usual, large values of γ imply long steps in agents movements, which are not significative if the mean opinion is conserved, but here, combined with the random selection of agents, add noise and could change the consensus opinion. However, after averaging several simulations starting with the same initial condition, we recover the consensus opinion obtained for smaller values of γ.

The agents are indexed by $1 \le i \le N$; in a single temporal step we take a random permutation *perm* of the full set of index, we match agents i and $j = perm(i)$, and both agents change opinions according to rule (1). If $perm(i) = i$, no changes occur.

Each agent has a temporal trajectory in the space of opinions, and the plots show several of this trajectories. A snapshot of agent location is taken after 10 temporal steps, which means that each agent performed about 20 interactions. When more than one final cluster is obtained, we take care that all of them are represented in the plot.

We have considered different classes of influence functions, and we describe below the results for each class.

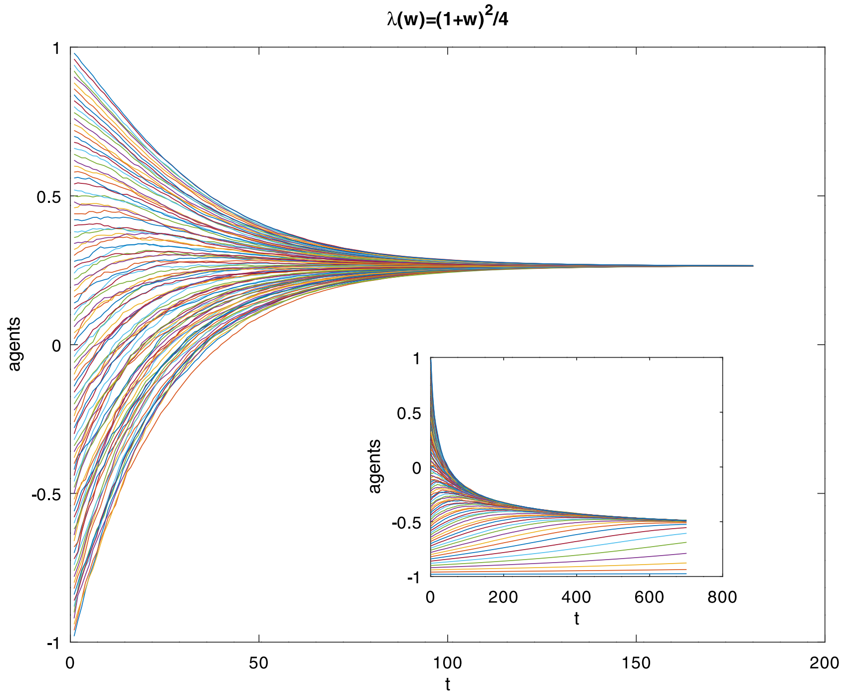

Fig. 1. Trajectories for the interaction rule (1) and 1000 agents, for $\lambda(w) = (1+w)^2/4$. Inset: the evolution of the same initial condition for $P(w) = (1+w)^2/4$ using the interaction rule (4).

4.1. Increasing attractiveness

Assuming that $\lambda(w_1) \le \lambda(w_2)$ whenever $w_1 \le w_2$, we have $m'(0) > 0$ since $m(0) = 0$. Since $m(t)$ cannot oscillate (if m' changes signs, for some $\hat{t}$ we have $m'(\hat{t}) = 0$, and m remains constant), and also $m(t) \le 1$, then there exists the limit m_∞ and it is positive. So, there exists a limit m_∞, and this value is given by (7).

In Fig. 1 we show the time evolution of a simulation of $N = 1000$ agents for $\lambda(w) = (1+w)^2/4$. Observe that the convergence occurs near the expected value $m_\infty \approx 0.26$. Also, in the inset, we depict the corresponding simulation for the same initial condition and $P(w) = (1+w)^2/4$ using the microscopic rule (4).

The consensus point with an increasing fitness is located on the right hand side of the interval. On the other hand, the dynamics given by Eqs. (4) has the opposite behavior, since agents with low values of the weight P are reluctant to change opinions and attract other agents to the left hand side of the interval. Let us stress the very slow convergence of the trajectories in the inset of Fig. 1, since near -1, $P(x)$ approaches zero quadratically.

4.2. Symmetric attractiveness

Assuming that $\lambda(-w) = \lambda(w)$ we get by symmetry that $\langle w\lambda(w)\rangle = 0$, and given an initial uniform distribution of agents, $f(w,t)$ is symmetric for any t since $f(-w,t)$ solves the same equation. Hence, if $f_t \to \delta_{m_\infty}$, we can expect that the consensus opinion is $m_\infty = 0$.

However, this equilibrium not necessarily is reached if $\lambda(0) = 0$. Intuitively, this opinion fails to attract new agents, and symmetry breaking phenomena appears. The population shifts to a different opinion, with no preferences between the positive or negative sides, defined only by the initial distribution of opinions and random fluctuations in the dynamics, which are amplified by Λ^{-1}.

The agent based simulations in Fig. 2, top row, left panel, show the convergence to consensus at the origin for $\lambda(w) = 1 - w^2$, where the fitness attains a maximum. Also, we show a typical simulation for $\lambda(w) = w^4$, an influence function which vanishes at the origin, on the top row, right panel. Observe the rupture of symmetry, and the slow convergence, which increases as the order of the zero at the origin increases.

Remark 4.1. Few remarks are in order:

- Adding a small constant ε to λ prevents this symmetry breaking, and a consensus at the origin is reached. Observe that Λ^{-1} grows linearly near the origin in this case, hence there is no amplification of the fluctuations.

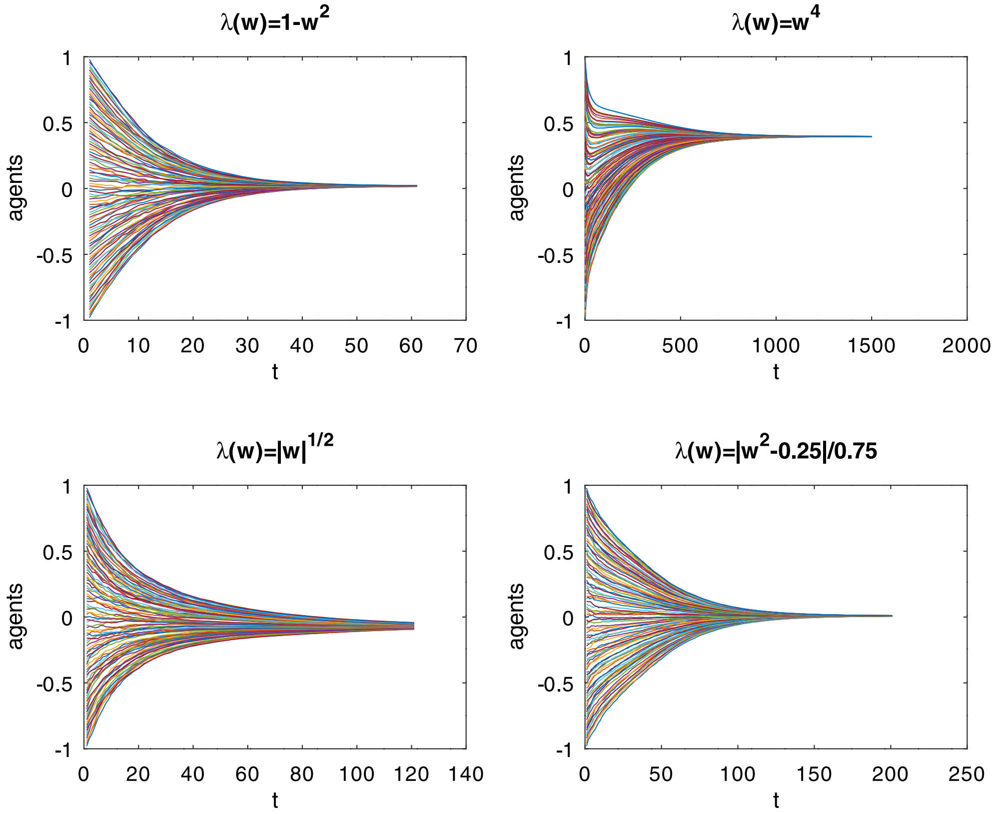

Fig. 2. Trajectories for the interaction rule (1) and 1000 agents, for different fitness functions. The right panel in the first file shows the rupture of symmetry for a high order zero of λ, and the slow convergence.

- The dynamics given by rule (4) with $P(w) = w^2$ or $P(w) = w^4$ is different, and in both cases the distribution converge slowly to a Dirac's mass at zero. Here, an agent located at zero acts as a stubborn or zealot which never changes opinion, attracting other agents to its own opinion. Moreover, the asymptotic behavior of agents is different, since two peaks are developed around the origin, and then they move towards zero.

4.3. Zeros of λ.

The zeros of λ have a high impact on the asymptotic distribution of agents since nobody is attracted by an opinion with zero fitness. The second row, left panel, in Fig. 2 shows the formation of consensus at $w = 0$ when we consider $\lambda(w) = |w|^{1/2}$. Despite the zero at the origin, Λ grows quickly in this case.

Then, we consider a symmetric function with $\lambda(-a) = \lambda(a) = 0$ for $a = 1/2$. As before, $f_\infty(w) = \delta_0$ is an equilibrium in our case, as in the second row, right panel, in Fig. 2, where we consider $\lambda = |w^2 - 0.25|/0.75$. A similar behavior is obtained for $\lambda = (|\,|w|-0.5|/0.5)^{1/2}$, in both cases we get convergence to $m_\infty = 0$.

However, the zeros at $\pm a$ can add other equilibria for the model given by the interaction rule (4). In Fig. 3 we present the simulations for $P(w) = |w^2 - 0.25|/0.75$, and $P(w) = (|\,|w|-0.5|/0.5)^{1/2}$. In the first figure, agents split in three groups, one at the origin, and the other ones at $\pm 1/2$. In the second one, we get consensus at the origin, although agents slow down their movement near the zeros and remain there for some time. Of course, any agent located at a zero will remain there.

5. Conclusions

In this work we have proposed an opinion formation model by considering the fitness or attractiveness of the opinions. Opinions with higher fitness will attract agents with more intensity than opinions with lower fitness. Each agent takes a step towards the other agent which is proportional to the fitness of the other agent opinion. Previous models in the literature considered a step towards the other agent being proportional to the fitness of the own opinion, and we have compared the results for both models.

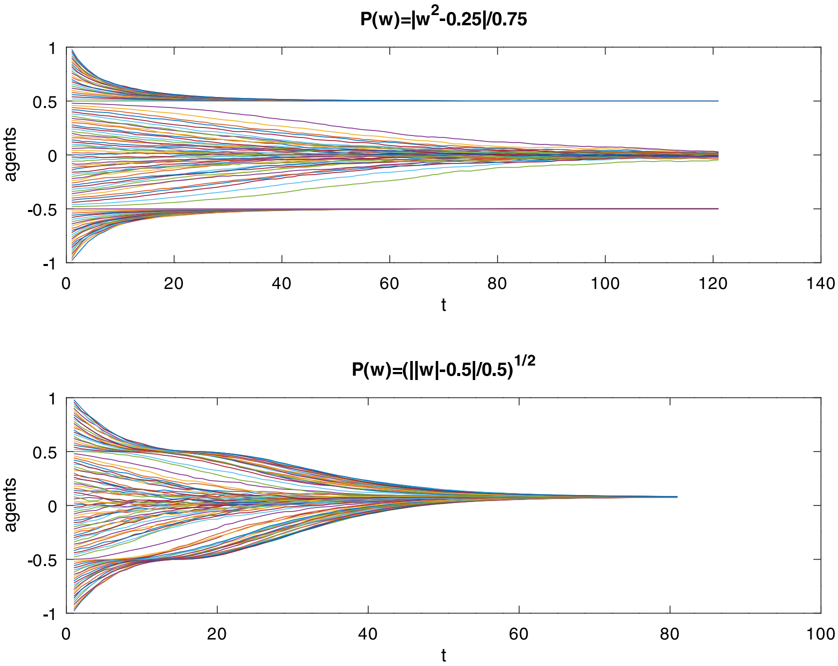

Fig. 3. Trajectories for the interaction rule (4) and 1000 agents, for $P(w) = |w^2 - 0.25|/0.75$ and $P(w) = (||w|-0.5|/0.5)^{1/2}$.

Using arguments from statistical physics and hydrodynamics we have derived a kinetic equation which is the weak formulation of a first order partial differential equation. Their coefficients depend on the integral and the expected value of the fitness function with respect to the density of agents. The convergence to a consensus is obtained assuming that agents have a tendency to compromise, and we have characterized the limiting opinion in terms of the initial expected value of a primitive of the fitness function.

We have shown with simulations the typical behavior of agents for increasing and for symmetric fitness functions. Also, the role of the zeros of the fitness functions was studied. A biased distribution towards the fittest opinions was obtained, as expected. A striking result was the formation of consensus at or close to a zero of the fitness function, despite the fact that a minimum is reached at this point. We can think of a population stuck in a dead end when faced to choose between identical alternatives, a well-known situation discussed in philosophy, the Buridan's ass paradox, although it can be found in Aristotle's treatise *On the Heavens*.

Finally, we do not provide here a rigorous proof of the convergence towards m_∞ using the interaction rule (1). We believe that such a proof can be obtained following the ideas in [27], although some additional hypothesis on λ seems to be necessary, like strict positivity to avoid the symmetry breaking observed near zeros of the influence function.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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